Environmental Protection and Dynamic Optimal Pollution Tax: Resource Modeling Approach

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Abstract

In this paper, we aim to analyze the dynamic structure of optimal tax for abating environmental pollution. For this purpose, we established a non-renewable resource model including environmental pollution stock as a state variable. With the optimality conditions of our model, we identified that the optimal time path of the shadow cost of environmental pollution stock is the same as that of the costate variable for environmental pollution stock. We derived this by applying the Theorem, Continuous Dependence on Initial Conditions (Coddington, E.A. & N. Levinston 1984, pp 22-27), to the optimal control problem. Hence, this result provides a theoretical basis in determining the magnitude of the optimal tax over time. In addition, we observed the roles of costate variables included in our model; the costate variable for resource stock can be decomposed between the scarcity effect and the cost effect, and the costate variable for environmental pollution stock is solely due to the disutility effect.

JEL Classification: Q 30

Keywords: Nonrenewable resource, Environmental pollution stock, Costate variables, The scarcity and cost effect, The disutility effect.

INTRODUCTION

There are few subjects in economics that have been discussed as extensively as the problem of environmental pollution. Following Pigou's initial insight on this subject (1932), a numerous of studies have been undertaken to design environmental policies for pollution abatement. In a static model analysis, it has been significantly suggested that if a regulatory agency imposes the value of marginal social damage incurred by environmental pollution as a Pigouvian tax, then the Pareto optimality in a society would be attained (Baumol 1972, Baumol and Oates 1988). In this analysis, the value of marginal social damage is denoted as the sum of the value of marginal disutility of consumers and the marginal cost of firms with respect to the increment of environmental pollution. On the other hand, as concerns about the spillover effect of pollution in economic growth process have increased (Mishan, 1969, Houghton et al. 1990) two approaches have been directed to observe the side effect of pollution on the optimal endogenous variables in the model. One approach has modified the optimal growth model to reflect environmental pollution (Forster 1973, Gruver 1976, Nordhaus 1993, Selden and Song 1995) and the other one has changed the nonrenewable resource model to include environmental pollution stock as a state variable (Forster 1980, Kolstad and Toman 2001).

The main result of the modified optimal growth model is that the rate of both the optimal consumption and capital at stationary state are lower than when environmental pollution is not considered (Forster 1973, Selden and Song 1995). A modified nonrenewable resource model has shown that the optimal extraction of resource is slowed in responding to the accumulation of pollution stock (Forster 1980, Kolstad and Toman 2001). Similar to the suggestion in a static model analysis, dynamic analyses considering environmental pollution have also proposed that levying an optimal tax reduces the rate of consumption of goods (or extraction of resource stock) over time; thereby, slows the accumulation of environmental pollution in the future (Nordhaus 1993, Kolstad and Toman, 2001).

However, these dynamic models are deficient in the theoretical analysis in determining the magnitude of optimal pollution tax over time. We, thus, aim to provide the theoretical basis of optimal tax over time by investigating the characteristics of costate variable for environmental pollution stock. Based on the optimality conditions of our model below, we identified that the shadow cost of environmental pollution stock at time t is equal to the costate variable for environmental pollution stock at that time. We derived this result by applying the Theorem, Continuous Dependence on Initial Conditions (Coddington, E.A. and N. Levinston 1984, pp 22-27), to the optimal control problem. Hence, if we identify the optimal time path of the shadow cost of environmental pollution stock, then we can elicit the appropriate information about the magnitude of optimal tax at each time. For this purpose, we first present a nonrenewable resource model with environmental pollution stock. Second, we discuss the role of the costate variable for both resource stock and environmental pollution stock, which are included in our model, respectively.

NONRENEWABLE RESOURCE MODEL WITH THE STOCK OF POLLUTION

The objective of this problem is to maximize the discounted present value of the net surplus stream subject to two constraints. These constraints are the laws of motion for both nonrenewable resource stock and the environmental pollution stock. The instantaneous utility function is assumed to be twice continuously differentiable, to increase at a decreasing rate with extraction of nonrenewable resource, x , and to decrease at an increasing rate with the environmental pollution stock, p. This properties implies that $u_x > 0,\ u_{xx} < 0$, and $u_p < 0,\quad u_{pp} > 0$. In addition, we assume that the cross partial derivative of the resource stock and environmental pollution stock is zero, i.e. $u_{xp} = 0$. The extraction cost function is written as $c(x(t), z(t))$ where $z(t)$ is the nonrenewable resource stock. We assume that extraction costs are increasing at an increasing rate with the rate of extraction, $c_x > 0, c_{xx} > 0$, and are increasing as the nonrenewable resource stock decreases, $c_z < 0$. The dynamic optimization problem is to maximize[3]

$$(1)\qquad W = \int_0^T e^{-\rho t}\{u(x(t), p(t)) - c(x(t), z(t))\}\, dt + e^{-\rho T} S(p(T))$$

subject to

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[3] In Forster's paper (1980), he maximized the objective function under given T instead of considering it as control parameter. In addition, he did not consider the terminal (scrap) value of environmental pollution stock in his model.

$$\frac{dz(t)}{dt} = -x(t)$$

$$\frac{dp(t)}{dt} = -\beta\ p(t) + \sigma\ x(t)$$

$$z(0) = z^0 \text{ given}, \quad p(0) = p^0 \text{ given}$$

$$x(t),\ z(t),\ p(t) > 0$$

In the law of motion for the stock of environmental pollution, the first term on the right-hand side denotes the natural rate of dissipation and decomposition of the existing environmental pollution stock and the second term indicates that the generation of new pollution is proportional to the extraction of the nonrenewable resource. Thus, β and σ are parameters with given values. ρ is the rate of time preference, and $S(\quad)$ is the terminal (scrap) value function at time T. The current value Hamiltonian with two state variables is

$$(2)\ H = u(x(t), p(t)) - c(x(t), z(t))\} + \psi_1(t)\{-x(t)\} + \psi_2(t)\{-\beta\ p(t) + \sigma\ x(t)\}$$

where $\psi_1(t)$ and $\psi_2(t)$ are the current value costate variables for nonrenewable resource stock and environmental pollution stock, respectively. We use the optimality theorem for the Hestenes Bolza problem as stated in Long and Vousden (1977, pp 11-34) in Theorem 1. In the terminology of this theorem, we have three control parameters. They are T, the stopping time for extractions, $z(T)$, the nonrenewable resource stock at that time, and $p(T)$, the environmental pollution stock at that time, In addition, we have a control variable, $x(t)$. The current value necessary conditions are

(3) $$u_x(x^*(t), p^*(t)) - c_x(x^*(t), z^*(t)\} = \psi_1^*(t) - \sigma\,\psi_2^*(t)$$

(4) $$\frac{d\psi_1^*(t)}{dt} = \rho\,\psi_1^*(t) + c_z(x^*(t), z^*(t))$$

(5) $$\frac{d\psi_2^*(t)}{dt} = (\rho+\beta)\,\psi_2^*(t) - u_p(x^*(t), p^*(t))$$

(6) $$\frac{dz^*(t)}{dt} = -x^*(t)$$

(7) $$\frac{dp^*(t)}{dt} = -\beta\; p^*(t) + \sigma\, x^*(t)$$

(8) $$z(0) = z^0, \quad p(0) = p^0$$

And the current value transversality conditions are

(9) $$\psi_1^*(T^*) \geq 0, \qquad \psi_1^*(T^*) z^*(T^*) = 0$$

(10) $$\psi_2^*(T^*) = S^{'}(p^*(T^*))$$

(11) $$u(x^*(T^*), p^*(T^*)) - c(x^*(T^*), z^*(T^*)) - \psi_1^*(T^*) x^*(T^*)$$

$$+\psi_2^*(T^*)\{-\beta\; p^*(T^*) + \sigma\, x^*(T^*)\} - \rho S(p^*(T^*)) = 0$$

As a marginal arbitrage equation, Equation (3) proposes that the marginal net surplus (benefit) of resource extraction is equal to the shadow value of resource stock adjusted to account for the shadow cost of additional pollution stock. The part of this sum on the right-hand side of Equation (3) exists because the marginal unit of the resource has value in other time periods, and the second part exists because the marginal unit of extraction causes pollution. In both cases the marginal units are valued at the value of a unit of the stock. This proposition implies that the optimal rate of resource extraction is lower due to the negative external effect of increasing the stock of environmental pollution than when the externality is ignored. This slows the accumulation of environmental pollution. Equation (4) shows the dynamic equation of the shadow value of resource stock, and is consistent with and different form of the Hotelling (1931)'s rule. Equations (3) to (5) give the information about how both economics and environmental pollution are interrelated in determining the optimal time path of endogenous variables in the model. Based on current value necessary as well as transversality conditions stated above, we begin to examine the characteristics of costate variables included in this model[4].

THE CHARACTERISTICS OF COSTATE VARIABLE

We discuss the costate variable for the stock of environmental pollution. To gain information about $\psi_2^*(t)$, we examine Equation (5). This differential equation with the terminal value, $\psi_2^*(T^*) = S'(p^*(T^*))$, has the solution (See Appendix for derivation).

$$\psi_2^*(t) = e^{-(\rho+\beta)(T^*-t)}\psi_2^*(T^*) + \int_t^{T^*} e^{-(\rho+\beta)(s-t)} u_p(x^*(s), p^*(s))\ ds \tag{12}$$

The optimal current value costate variable, $\psi_2^*(t)$, gives the time path of the shadow cost of the environmental pollution stock. The costate variable for the stock of

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4 Lyon (1999) analyzed the costate variables for nonrenewable and renewable resource stock in separated models, respectively. Our model is a good example to contrast the role of costate variable for exhaustible economic good with that of economic bad at once.

environmental pollution, $\psi_2^*(t)$, is the current value of the change of the solution of Equation (1) per unit change in environmental pollution stock at time *t*. For time zero, this can be stated as $\frac{\partial W^*}{\partial p^0} = \psi_2^*(0)$ [5]. Below, we derive this statement. To do this rigorously, let us define $p^*(t) = \phi^*(t, p^0)$ be the optimal time path of *p* given the initial condition is p^0 . With this we can write the optimal solution of Equation (1) as

$$(13) \qquad W^*(p^0) = \int_0^{T^*} e^{-\rho t}\{u(x^*(t), \phi^*(t, p^0)) - c(x^*(t), z^*(t)\}\, dt + e^{-\rho T^*} S(\phi^*(T^*, p^0)$$

Differentiation Equation (12) with respect to p^0 yields

(14)

$$\frac{\partial W^*(p^0)}{\partial p^0} = \int_0^{T^*} e^{-\rho t} u_{p^*}(x^*(t), \phi^*(t, p^0)) \frac{\partial \phi^*(t, p^0)}{\partial p^0}\, dt + e^{-\rho T^*} S'(\phi^*(T^*, p^0)) \frac{\partial \phi^*(t, p^0)}{\partial p^0}$$

We use the Theorem, Continuous Dependence on Initial Conditions (CDIC), to get the information about $\phi^*_{p^0}(t, p^0)$. Write Equation (7) as

$$\frac{dp^*(t)}{dt} = f(t, p^*(t)) \qquad \text{with} \quad p^*(0) = p^0$$

By the theorem CDIC, $\phi^*_{p^0}(t, p^0)$ satisfies the initial value problem

$$(15) \qquad \frac{\partial \phi^*_{p^0}(t, p^0)}{\partial t} = f_{p^*}(t, p^*) \phi^*_{p^0}(t, p^0) \quad \text{with} \quad \phi^*_{p^0}(0, p^0) = 1$$

This initial condition, $\phi^*_{p^0}(0, p^0) = 1$, exists so that $p^*(0) = \phi^*(0, p^0)$ will change at the same rate as p^0, keeping $p^*(0) = p^0$. To gain some feel for this equation, note that a

[5] This is the common statement that the costate variable is the shadow value of the state variable.

solution of Equation (7), $\phi^*(t, p^0)$, means

$$\frac{\partial \phi^*(t, p^0)}{\partial t} = f(t, \phi^*(t, p^0))$$

Hence,

$$\frac{\partial(\partial \phi^* / \partial t)}{\partial p^0} = f_{P^*}(t, \phi^*(t, p^0))\phi^*_{p^0}(t, p^0)$$

and by Young's Theorem,

$$\frac{\partial(\partial \phi^* / \partial t)}{\partial p^0} = \frac{\partial \phi^*_{p^0}}{\partial t}$$

Combining these yields Equation (15)

From Equation (5), $f_{p^*}(t, p^*(t)) = -\beta$,

Thus

(16) $$\frac{\partial \phi^*_{p^0}(t, p^0)}{\partial t} = -\beta \phi^*_{p^0}(t, p^0) \qquad \text{with} \quad \phi^*_{p^0}(0, p^0) = 1$$

The solution of this initial value problem is

(17) $$\phi^*_{p^0}(t, p^0) = e^{-\beta t}$$

Inserting Equation (10), and (17) into the right-hand side of Equation (14), and simplifying yields

$$\frac{\partial W^*}{\partial p^0} = \psi^*_2(0).$$

This result shows that the costate variable for environmental pollution stock is the present value of cost stream of the marginal unit of environmental pollution stock. If there is private ownership of the exhaustible resource and the sellers are price takers, then the value of the resource will be competed into the market price of x; hence we can generate the equality in Equation (3) by imposing an optimal tax of $\sigma\psi_2^*(t)$ per unit of x. If this resource is fossil fuels then the tax results in reduced the rate of fossil fuel extraction and thereby a reduction in the accumulation of environmental pollution.

Furthermore, we discuss the role of the optimal current value of costate variable for the resource stock, $\psi_1^*(t)$. According to Lyon(1999) that analyzed the role of costate variable for the stock of nonrenewable resource, the differential equation (4) with terminal value, $\psi_1^*(T^*)$, has the solution

$$\psi_1^*(t) = e^{-\rho(T^*-t)}\psi_1^*(T^*) - \int_t^{T^*} e^{-\rho(s-t)} c_z(x^*(s), z^*(s))\, ds$$

This equation shows the shadow value of the resource stock at time *t*, and it implies the current value rate of change in the solution value of Equation (1) per unit change in resource stock in time *t*. The shadow value of resource stock can be decomposed into two parts such as

$$e^{-\rho(T^*-t)}\psi_1^*(T^*), \qquad \text{as the Scarcity Effect}$$

and $$-\int_t^{T^*} e^{-\rho(s-t)} c_z(x^*(s), z^*(s))\, ds \qquad \text{as the Cost Effect}$$

In addition, we can separate the shadow cost of environmental pollution stock into two components according to Equation (12):

$$e^{-(\rho+\beta)(T^*-t)}\psi_2^*(T^*) \qquad \text{as the Undesirable Abundance Effect}$$

and $$\int_t^{T^*} e^{-(\rho+\beta)(s-t)} u_p(x^*(s), p^*(s)) \; ds \qquad \text{as the Disutility Effect}$$

The undesirable abundance effect is simply that the terminal value of environmental pollution stock discounted to the current time t, and the disutility effect shows the present value of the increment of disutility associated with the marginal unit of environmental pollution stock increase. If we investigate this variable furthermore, however, we identify that there are not two effects, but only one effect. They are both the present value of the disutility effect. If the optimal stopping time comes from the nonrenewable resource and nonrenewable resource is only the source of environmental pollution, the stock of environmental pollution will not be accumulated any more after the optimal stopping time, and it will gradually disappear. Consequently, the stock of environmental pollution is negligible when the time is extremely larger than the optimal stopping time. In this sense, the terminal value of environmental pollution stock is

$$\text{(20)} \qquad \psi_2^*(T^*) = \int_{T^*}^{\infty} e^{-(\rho+\beta)(s-T^*)} u_p(0, p^*(s)) \; ds$$

which is the disutility effect. Hence, in this case the undesirable effect is the disutility effect. As a result, we conclude that the costate variable for the stock of environmental pollution is solely due to the disutility effect. This is the difference from the costate variable for resource stock, which is decomposed between the scarcity effect and the cost effect.

SUMMARY

We presented a nonrenewable resource model including environmental pollution as a state variable. Based on the optimality conditions of our model, we have shown that the optimal time path of the shadow value of environmental pollution stock is the same as that of the costate variable for environmental pollution stock. Thus, if a regulatory agency imposes some portion of the costate variable for environmental pollution stock as optimal tax over time, it will reduce the rate of resource extraction, and thereby slow the accumulation of environmental pollution. In addition, we have discussed the characteristics of costate variables for the stock of environmental pollution included in our model as a state variable. We observed that the costate variable the stock of environmental pollution is solely due to the disutility effect.

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APPENDIX

The proof of Equation (12)

The Equation (5) is

$$\frac{d\psi_2^*(t)}{dt} = (\rho+\beta)\,\psi_2^*(t) - u_p(x^*(t), p^*(t))$$

with $\psi_2^*(T^*)$ given by current value transversality condition. This can be arranged into

$$\frac{d\psi_2^*(t)}{dt} - (\rho+\beta)\,\psi_2^*(t) = -u_p(x^*(t), p^*(t))$$

which is a linear first order differential equation with a variable coefficient and a variable term. Then, the general solution of this differential equation can be written

$$(A-1) \qquad \psi_2^*(t) = e^{(\rho+\beta)t}(A - \int e^{-(\rho+\beta)t} u_p(x^*(t), p^*(t)))\ dt$$

where A is the constant of integration.

Let us define

$$G(t) := \int e^{-(\rho+\beta)t} u_p(x^*(t), p^*(t))\ dt$$

Thus, Equation $(A-1)$ can be written

$$\psi_2^*(t) = e^{(\rho+\beta)t}(A - G(t))$$

and $\qquad \psi_2^*(T^*) = e^{(\rho+\beta)T^*}(A - G(T^*))$

Thus,

$$A = e^{-(\rho+\beta)T^*}\psi_1^*(T^*) + G(T^*)$$

Therefore,

$$\psi_2^*(t) = e^{-(\rho+\beta)(T^*-t)}\psi_2^*(T^*) + e^{(\rho+\beta)t}(G(T^*) - G(t))$$

$$\psi_2^*(t) = e^{-(\rho+\beta)(T^*-t)}\psi_2^*(T^*) + \int_t^{T^*} e^{-(\rho+\beta)(s-t)} u_p(x^*(s), p^*(s))\ ds$$